FUNCTION

* All the basics of the function like what is function, types of function and algebraic operations have been studied in earlier classes (means class XIth). Now,

* Kinds of Functions :-

1. **One - One functions (Injection) :-** A function $f: A \to B$ is said to be a One-one function or an injection if different elements of A have different images in B.

Thus, $f: A \to B$ is one-one

$\Leftrightarrow \quad a \neq b \Rightarrow f(a) \neq f(b) \quad \forall\ a, b \in A$

$\Leftrightarrow \quad f(a) = f(b) \Rightarrow a = b \quad \forall\ a, b \in A.$

eg :- If $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5, 6\}$ and $f: A \to B$ be a function defined by $f(x) = x + 2 \ \forall\ x \in A$. Is it a One-one function?

Soln :- From Question,

$$f = \{(1,3), (2,4), (3,5), (4,6)\}.$$

Clearly different element of A has different images under function f.

So, $f: A \to B$ is a one-one function / injection.

[Note :- Let $f: A \to B$ and let $x, y \in A$. Then $x = y \Rightarrow f(x) = f(y)$ is always true from the definition. But $f(x) = f(y) \Rightarrow x = y$ is true only when f is one-one.]

eg :- Find whether the following functions are one-one or not:

(a) $f: R \to R$ given by $f(x) = x^3 + 2 \quad \forall\ x \in R$

(b) $f: Z \to Z$ given by $f(x) = x^2 + 1 \quad \forall\ x \in Z$

Soln :- Let x, y be two arbitrary elements of R (domain of f) such that $f(x) = f(y)$, then,

(a) $f(x) = f(y) \Rightarrow x^3 + 2 = y^3 + 2 \Rightarrow x^3 = y^3 \Rightarrow x = y$. True.

(1) ②

(ii) $f(x) = f(y)$

$\Rightarrow x^2+1 = y^2+1 \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$. False

[Note:- If A and B are two sets having m and n elements respectively such that $m \le n$, then total no. of one-one functions from A to B is ${}^nC_m \times m!$.]

2. Many-One function:- A function $f: A \to B$ is said to be many-one function if two or more elements of set A have the same images in B.

Thus, $f: A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \ne y$ but $f(x) = f(y)$.

→ In other words, $f: A \to B$ is a many-one function if it is not a one-one function.

eg:- If $A = \{-1, 1, -2, 2\}$, $B = \{1, 4, 9, 16\}$ and $f: A \to B$ given by $f(x) = x^2$. then the the function is whether one-one/many-one function?

Soln:- From Question:-

$$f = \{(-1, 1), (1, 1), (-2, 4), (2, 4)\}$$

clearly 1 and -1 have the same images. Similarly for 2.

So, it is a many-one function.

eg:- Show that the function $f: Z \to Z$ defined by $f(x) = x^2+x\ \forall\ x \in Z$, is a many-one function.

Soln:- Let $x, y \in Z$. then,

$f(x) = f(y)$

$\Rightarrow x^2+x = y^2+y$

$\Rightarrow (x^2-y^2) + (x-y) = 0$.

$\Rightarrow (x-y)(x+y+1) = 0$

$x = y$ or $y = -x-1$.

Since $f(x) = f(y)$ doesn't provide the unique soln. This means $x \ne y$.

hence, f is a many-one function.

3) Onto Function:- (Surjection):- A function $f: A \to B$ is said to be an Onto function or a Surjection if every element of B is the image of some element of A i.e $f(A) = B$ or range of f is the Co-domain of f.

→ Thus $f: A \to B$ is a Surjection if for each $b \in B$, there exists $a \in A$ such that $f(a) = b$.

* Into Function:- A function $f: A \to B$ is said to be an Into function if there exists an element in B having no pre-image in A.

→ In other words, $f: A \to B$ is an into function if it is not an Onto function.

eg:- If $A = \{-1, 1, 2, -2\}$, $B = \{1, 4\}$ and $f: A \to B$ be a function defined by $f(x) = x^2$, then show that f is an Onto function.

Solⁿ:- from Question,

$$f = \{(-1, 1), (1, 1), (2, 4), (-2, 4)\}.$$

Clearly, every element of B has pre-images in A. hence $f: A \to B$ is an Onto function.

eg:- Discuss Surjectivity of the following functions:-

(i) $f: R \to R$ given by $f(x) = x^3 + 2$ $\forall$ $x \in R$

(ii) $f: Z \to Z$ given by $f(x) = 3x + 2$ $\forall$ $x \in Z$

(iii) $f: R \to R$ given by $f(x) = x^2 + 2$ $\forall$ $x \in R$

Solⁿ:- Let y be any element of R. then,

① $f(x) = y = x^3 + 2 \Rightarrow x = (y-2)^{1/3}$

Clearly for all $y \in R$, $(y-2)^{1/3}$ is a real no.

Thus for all $y \in R$, there exists $x = (y-2)^{1/3}$ in R such that $f(x) = x^3 + 2 = y$. Hence $f: R \to R$ is an onto function.

(ii) $f(x) = y = 3x + 2 \Rightarrow x = \frac{y-2}{3}$. then if $y = 0$, $x \neq \frac{-2}{3} \notin Z$

So, f is not an onto function.

$f(x) = x^2+2$. Clearly $x^2+2 \geq 2$ so, negative real no. in R (domain) don't have pre-images in R. hence f is not an onto function.

4) One-One Onto Function :- (Bijection) :-

A function $f: A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f: A \to B$ is a bijection, if

(i) it is one-one i.e. $f(x) = f(y) \Rightarrow x = y \ \forall x, y \in A$.

(ii) it is onto i.e. $\forall y \in B$, there exists $x \in A$ such that $f(x) = y$.

eg:- Prove that the function $f: Q \to Q$ given by $f(x) = 2x-3 \ \forall x \in Q$ is a bijection.

Soln:- Let x, y be two arbitrary elements in Q. then,

$$f(x) = f(y) \Rightarrow 2x-3 = 2y-3 \Rightarrow x = y.$$

Thus $f(x) = f(y) \Rightarrow x = y. \ \forall x, y \in Q$

So, f is an injection.

Now Let y be any arbitrary element of Q. then,

$$f(x) = y = 2x-3 \Rightarrow x = \frac{y+3}{2}$$

Clearly for all $y \in Q$, $x = \frac{y+3}{2} \in Q$.

$\therefore$ f is a surjection.

hence $f: Q \to Q$ is a bijection.

eg:- Let $A = \{x \in R, -1 \leq x \leq 1\} = B$ show that $f: A \to B$ given by $f(x) = x|x|$ is a bijection.

Soln:- Let x, y be two arbitrary element in A, then

$$x \neq y \Rightarrow x|x| \neq y|y| \Rightarrow f(x) \neq f(y)$$

So f is an injective map.

Surjectivity :-

we have, $f(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

for $0 \leq x \leq 1$ $f(x) = x^2$ and for $-1 \leq x < 0$ $f(x) = -x^2$ for every value of y, there is a pre-image in A. hence it is a bijection

[Note:- It follows from the above discussion that if A and B are two finite sets and $f: A \to B$ is a function, then

(i) f is an injection $\Rightarrow n(A) \leq n(B)$

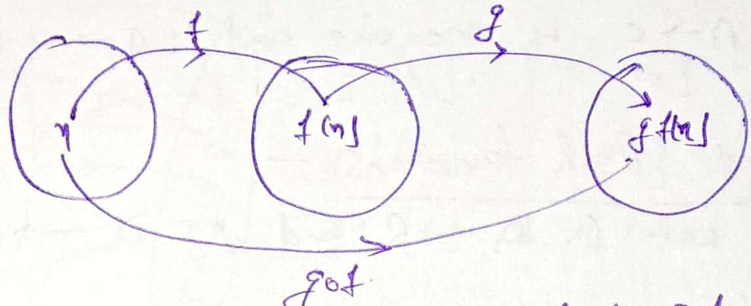
(ii) f is a surjection $\Rightarrow n(A) \geq n(B)$

(iii) f is a bijection $\Rightarrow n(A) = n(B)$.]

* Composition of functions:-

Let $f: A \to B$ and $g: B \to C$ be two functions. Then a function $gof: A \to C$ defined by

$gof(x) = g(f(x)), \forall x \in A$ is called the Composition of f and g.



Outcomes:-

1. It is evident from the definition that gof is defined only if for each $x \in A$, $f(x)$ is an element of g so that we can take its g-image. Hence for the composition gof to exist, the range of f must be a subset of the domain of g.
2. Similarly, fog exists if range of g is a subset of domain of f.

* Properties of Composition of functions:-

1. The Composition of functions is not commutative i.e. $fog \neq gof$.

2. The composition of function is associative i.e. if f, g, h are three functions such that $(fog)oh$ and $fo(goh)$ exist, then
$(fog)oh = fo(goh)$

Proof:- $h: A \to B$, $f: C \to D$, $g: B \to C$ $(fog): B \to D$ & $(fog)oh: A \to D$ Similarly, $(goh): A \to C$ $fo(goh): A \to D$ etc.

3. The Composition of two ~~functions~~ bijections is a bijection i.e. if f and g are two bijections, then gof is also a bijection.

4.) Let $f: A \to B$. ~~then~~ $f \circ I_A = I_B \circ f = f$ i.e. the composition of any function with the identity function is the function itself.

5.) Let $f: A \to B$, $g: B \to A$ be two functions such that $g \circ f = I_A$. then, f is an injection and g is a surjection.

6.) Let $f: A \to B$, $g: B \to A$ be two functions such that $f \circ g = I_B$. then f is a surjection and g is an injection.

7.) Let $f: A \to B$ and $g: B \to C$ be two functions. then,

(i) $g \circ f: A \to C$ is onto $\Rightarrow g: B \to C$ is onto

(ii) $g \circ f: A \to C$ is one-one $\Rightarrow f: A \to B$ is one-one

(iii) $g \circ f: A \to C$ is onto and $g: B \to C$ is one-one $\Rightarrow f: A \to B$ is onto.

(iv) $g \circ f: A \to C$ is one-one and $f: A \to B$ is onto $\Rightarrow g: B \to C$ is one-one.

* Composition of Real functions:—

Let $f: D_1 \to R$ and $g: D_2 \to R$ be two real function. then,

$$g \circ f : X = \{ x \in D_1 : f(x) \in D_2 \} \to R$$

and,

$$f \circ g : Y = \{ x \in D_2 : g(x) \in D_1 \} \to R$$

are defined as

$$g \circ f (x) = g(f(x)) \ \forall\ x \in X \quad \text{and} \quad f \circ g (x) = f(g(x)) \ \forall\ x \in Y.$$

[Note:- (i) If Range $(f) \subseteq$ Domain (g), then $g \circ f : D_1 \to R$ and If Range $(g) \subseteq$ Domain (f), then $f \circ g : D_2 \to R$

(ii) For any two real functions f and g, it may be possible that $g \circ f$ exists but $f \circ g$ does not. In some cases, even if both exist, they may not be equal.

(iii) If Range $(f) \cap$ Domain $(g) = \phi$, then $g \circ f$ doesn't exist. In other words $g \circ f$ exists if Range$(f) \cap$ Domain $(g) \neq \phi$.

(iv) If f and g are bijection, the $f \circ g$ and $g \circ f$ both are bijections

(v) If $f: R \to R$ and $g: R \to R$ be ~~two~~ real functions, then $f \circ g$ and $g \circ f$ both exist.

* Inverse of an element :-

Let A and B be two sets and $f: A \to B$ be a mapping.
then if $a \in A$ and $b \in B$.

$f(a) = b$. and $\boxed{f^{-1}(b) = a.}$

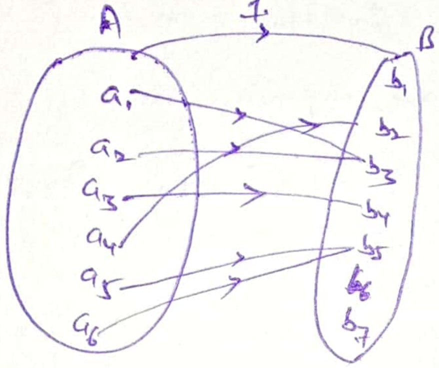
→ The inverse of an element under a function may consist of single/more element, it all depend on what type of function is given: If it is one-one → Inverse will have single

" " many-one → " " " many

" " onto → " " " single/many.

" " into → " " " No/single/many.

eg:-



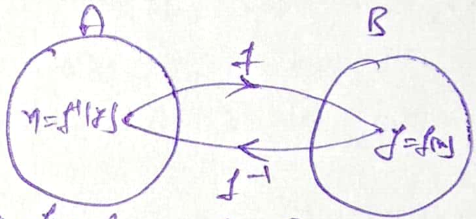
$f^{-1}(b_3) = \{a_1, a_2\}$.

$f^{-1}(b_4) = \{a_3\}$

$f^{-1}(b_1) = \phi$.

Similarly all.

* Inverse of a function :-



Let $f: A \to B$ be a bijection. Then a function $g: B \to A$ which associates each element ~~of~~ $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called ~~the~~ inverse of 'f'.

i.e $f(x) = y \Leftrightarrow g(y) = x$.

The inverse of 'f' is generally denoted by f^{-1}.

Thus, if $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is such that

$$f(x) = y \Leftrightarrow f^{-1}(y) = x.$$

eg:- Let $f: R \to R$ be defined by $f(x) = 3x - 7$. Show that ~~if~~ 'f' is invertible and hence f^{-1}.

Sol:- Injection:- Let $x, y \in R$, then,

$f(x) = f(y)$

$\Rightarrow 3x - 7 = 3y - 7$

$\therefore x = y$ True

Surjection:- Let y be any arbitrary element of R, then,

$y = 3x - 7 \Rightarrow x = \frac{y+7}{3} \in R$

True $\forall y \in R$.

Now: Let $f(x) = y$.

$y = 3x - 7$

$\Rightarrow x = \frac{y+7}{3}$ ∴ $f^{-1}(x) = \frac{x+7}{3}$ $f^{-1}: R \to R$.

* Properties of inverse of a function:-

1) The inverse of a bijection is unique

2.) The inverse of a bijection is also a bijection.

3.) If $f: A \to B$ is a bijection and $g: B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity functions on the set A and B respectively.

4.) If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is a bijection and $(gof)^{-1} = f^{-1} o g^{-1}$.

5.) Let $f: A \to B$ and $g: B \to A$ be two functions such that $gof = I_A$ and $fog = I_B$. then f and g are bijection and $g = f^{-1}$.

6.) Let $f: A \to B$ be an invertible function. then the inverse of f^{-1} is f; i.e. $(f^{-1})^{-1} = f$.

7.) Sometime $f: A \to B$ is one-one but not onto. In such a case 'f' is not invertible. But, $f: A \to Range(f)$ is both one-one and onto. So, it is invertible and its inverse can be found.

eg:-

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